EIGENVECTORS FOR A RANDOM WALK ON A HYPERPLANE ARRANGEMENT

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Abstract. We find explicit eigenvectors for the transition matrix of the Bidegare-Hanlon-Rockmore random walk, from [1]. This is accomplished by using Brown and Diaconis' analysis in [3] of the stationary distribution, together with some combinatorics of functions on the face lattice of a hyperplane arrangement, due to Gel′fand and Varchenko [10].

1. Introduction

In 1999, Bidegare, Hanlon and Rockmore [1] introduced a simultaneous generalization of several well-studied discrete Markov chains. Let $\mathcal{A}$ be an arrangement of n linear hyperplanes in $W = \mathbb{R}^\ell$, and let $\mathcal{C}$ denote the set of chambers: i.e., the connected components of the complement of $\mathcal{A}$ in W. They construct a random walk on $\mathcal{C}$, which we will follow [3] in calling the BHR random walk, by means of the face product, defined below. By choosing the hyperplane arrangement suitably, one obtains as special cases the Tsetlin library ("move-to-front rule"), Ehrenfests' urn, and various card-shuffling models [1, 3].

A key insight in this construction and main result of [1] is that the eigenvalues of the transition matrix can be expressed simply in terms of the combinatorics of the arrangement $\mathcal{A}$. This is useful for bounding the rate of convergence to the stationary distribution: Brown and Diaconis make this analysis in [3], show that the transition matrix is diagonalizable, and give an explicit description of the random walk's stationary distribution.

Subsequently Brown [2] generalized the BHR random walk to the setting of certain semigroup algebras, showing in particular that the diagonalization result held there as well. He describes projection operators onto each eigenspace, which in principle provides a description of the eigenvectors of the random walk's transition matrix. However, the general expression is necessarily somewhat complicated.

The purpose of this paper, then, is to provide a relatively straightforward description of the original BHR random walk's eigenvectors. It turns out that the combinatorial Heaviside functions of Gel′fand and Varchenko [10] behave well with respect to the face product and the BHR random walk, so we highlight their role in this problem. Using the description of the stationary distribution from [3], the main result here,

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Theorem 3.6, describes a spanning set for each eigenspace in terms of flags in the intersection lattice.

2. Background and notation

2.1. **The face algebra.** Let $\mathcal{A}$ be an arrangement of n hyperplanes in $\mathbb{R}^\ell$. We will assume throughout that $\mathcal{A}$ is central and essential: that is, the intersection of the hyperplanes $\mathcal{A}$ equals the origin in $\mathbb{R}^\ell$. We will follow the notational conventions of the standard reference for hyperplane arrangements, [6]. In particular, let $L(\mathcal{A})$ denote the lattice of intersections, ordered by reverse inclusion, and $\mathcal{F} = \mathcal{F}(\mathcal{A})$ the face semilattice, also ordered by reverse inclusion. For $0 \leq p \leq \ell$, let $L_p(\mathcal{A})$ and $\mathcal{F}_p(\mathcal{A})$ denote the subspaces and faces, respectively, of codimension p. In particular, the set of chambers $\mathcal{C} = \mathcal{C}(\mathcal{A}) = \mathcal{F}_0(\mathcal{A})$.

For a face $F \in \mathcal{F}$, let $|F|$ denote the smallest subspace in $L(\mathcal{A})$ containing F. Recall that $\mathcal{A}^X$ denotes the arrangement in X of hyperplanes $\{H \cap X \colon H \in \mathcal{A}, X \not\subseteq H\}$, whenever $X \in L(\mathcal{A})$, and $\mathcal{A}_X$ denotes the subarrangement $\{H \in \mathcal{A} \colon X \subseteq H\}$. We will identify faces (and chambers) of $\mathcal{A}^X$ with faces $F \in \mathcal{F}(\mathcal{A})$ for which $|F| \geq X$ (and $|F| = X$, respectively):

$$\mathcal{F}(\mathcal{A}^X) \cong \{F \in \mathcal{F}(\mathcal{A}) \colon |F| \geq X\}. \tag{2.1}$$

For hyperplanes $H \in \mathcal{A}$, let F_H be $0, \pm 1$ depending on whether F is contained in H, on the positive side, or the negative side, respectively. We will abbreviate the values of F_H with $0, +, -$. A face F is uniquely determined by the sign sequence $(F_H)_{H \in \mathcal{A}}$.

We recall the face product of two faces F and G can be described by its sign sequence:

$$(FG)_H = \begin{cases} F_H & \text{if } F_H \neq 0; \\ G_H & \text{otherwise.} \end{cases} \tag{2.2}$$

From the definition, $FF = F$ for any F, and $FGF = FG$ for any faces F, G (the "deletion property": see [2].)

For a fixed arrangement $\mathcal{A}$, let $A = R[\mathcal{F}]$ denote its face algebra, introduced in [3]. Additively, A is the free R-module on $\mathcal{F}$, and multiplication is given by linearly extending the face product. Following [3], let $V(\mathcal{A})$ denote the free R-module on $\mathcal{C}(\mathcal{A})$: then the face product makes $V(\mathcal{A})$ a left A-module. More generally, following [3, Section 5C], we can make $V(\mathcal{A}^X)$ a left A-module for any $X \in L(\mathcal{A})$: for $F \in \mathcal{F}$ and $G \in \mathcal{C}(\mathcal{A}^X)$, set

$$FG = \begin{cases} FG & \text{if } |F| \geq X; \\ 0 & \text{otherwise.} \end{cases} \tag{2.3}$$

2.2. **The zonotope of a real arrangement.** As Brown and Diaconis [3] observe, it is useful in this setting to consider the zonotope of a real arrangement. Let f_H denote a linear equation defining the hyperplane H, for each $H \in \mathcal{A}$. The zonotope $Z_{\mathcal{A}}$ is, by definition, the Minkowski sum of intervals,

$$Z_{\mathcal{A}} = \sum_{H \in \mathcal{A}} [-f_H, f_H].$$

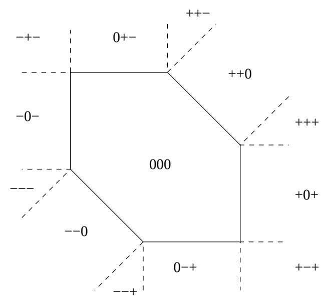


FIGURE 1. Zonotope for $f_1 = x$, $f_2 = y$, $f_3 = x - y$.

This is a polytope in W^*: we refer to [12] for details. Its most relevant feature here is that the face poset of $Z_{\mathcal{A}}$ (ordered by inclusion) is isomorphic to $\mathcal{F}(\mathcal{A})$: the isomorphism arises by identifying the outer normal fan of the zonotope with the arrangement. If F is a face of $\mathcal{A}$, then, let $\hat{F}$ denote the corresponding face of the zonotope. Each face $\hat{F}$ is an affine translate of a zonotope of a closed subarrangement: for $F \in \mathcal{F}$, let $X = |F|$ and $v_F = \sum_{H \in \mathcal{A}} (F_H) f_H$. Then $\hat{F} = v_F + Z_{\mathcal{A}_X}$: see, for example, discussion in [5].

For the sake of intuition, we indicate one way to visualize the face product in the zonotope world.

Definition 2.3. Suppose P is a polytope in W^* and $v \in W^*$ is nonzero. Define a map $r_{P,v} \colon P + [-v, v] \to P + v$ as follows. If $p \in P + [-v, v]$, let λ be maximal for which $p = x + \lambda v$, where $x \in P$ and $\lambda \leq 1$. Set $r_{P,v}(p) = x + v$. Clearly $r_{P,v}$ is piecewise-linear and fixes $P + v$ pointwise.

Proposition 2.4. *For each face F of $\mathcal{A}$, there is a piecewise-linear retract p_F of $Z_{\mathcal{A}}$ onto $\hat{F}$, with $p_F(\hat{G}) = \widehat{FG}$, for all $G \in \mathcal{F}(\mathcal{A})$.*

Proof. We will write p_F as the composition of the retracts from Definition 2.3, one for each hyperplane of $\mathcal{A}$ not containing F.

Again, let $X = |F|$. Let $S_1 = \{f_H \colon F_H = 0\}$ and $S_2 = \{(F_H) f_H \colon F_H \neq 0\}$. Then

$$\begin{aligned} Z_{\mathcal{A}} &= \sum_{v \in S_1} [-v, v] + \sum_{v \in S_2} [-v, v] \\ &= Z_{\mathcal{A}_X} + \sum_{v \in S_2} [-v, v], \end{aligned}$$

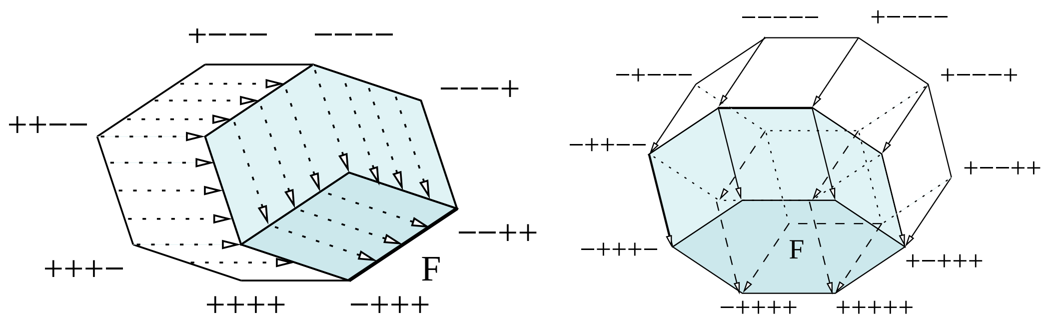
and $v_F = \sum_{v \in S_2} v$. Put the elements of S_2 in any order, writing $S_2 = \{v_1, v_2, \ldots, v_k\}$. Let $P_0 = Z_{\mathcal{A}_X} + v_F$, and let $P_i = P_{i-1} + [-2v_i, 0]$ for $1 \leq i \leq k$. Then

$$\hat{F} = P_0 \subseteq P_1 \subseteq \cdots \subseteq P_k = Z_{\mathcal{A}},$$

and the following composite collapses along the vectors in S_2, one at a time:

$$p_F = r_{P_0 - v_1, v_1} \circ r_{P_1 - v_2, v_2} \circ \cdots \circ r_{P_{k-1} - v_k, v_k} \colon Z_{\mathcal{A}} \to \hat{F}.$$

By construction, this is a piecewise-linear retract onto $\hat{F}$.



(a) p_F with $F = -0{+}{+}$ (b) p_F with $F = 00{+}{+}0$

FIGURE 2. Retracts onto faces of zonotopes of dimension 2 and 3

To see that $p_F(\hat{G}) = \widehat{FG}$ for any face G, let $S \in (\pm 1)^{\mathcal{A}}$ be an arbitrary sequence of nonzero signs, and define $v_S = \sum_{H \in \mathcal{A}} S_H f_H$. Then $r_{P_{i-1}-v_i, v_i}(v_S) = v_{S'}$, where

$$(S')_H = \begin{cases} S_H & \text{if } f_H \neq \pm v_i; \\ c & \text{if } f_H = c v_i. \end{cases}$$

It follows that $p_F(\hat{C}) = \widehat{FC}$ for chambers $C \in \mathcal{C}(\mathcal{A})$. By writing an arbitrary face $\hat{G}$ as the convex hull of the vertices $\hat{C}$ for which $C \leq G$, one then obtains $p_F(\hat{G}) = \widehat{FG}$ for all G. □

Note that the map p_F is not uniquely defined, and depends on a choice of order. See Figure 2 for examples.

2.5. **The BHR random walk.** Let $\mathcal{A}$ be an arrangement, and $w \colon \mathcal{F} \to \mathbb{R}$ a discrete probability distribution on the faces of $\mathcal{A}$. The random walk introduced in [1] is a random walk on chambers, taking $C \in \mathcal{C}$ to FC with probability w_F. Its transition matrix $K = K_{\mathcal{A}}$ can be regarded as a linear endomorphism of the vector space with basis $\mathcal{C}$, given by

$$K(C) = \sum_{F \in \mathcal{F}} w_F FC. \tag{2.4}$$

An eigenvector of K with eigenvalue 1 gives a stationary distribution for the random walk. Brown and Diaconis [3, Theorem 2] find that, with an assumption of nondegeneracy, the eigenspace is 1-dimensional, which is to say the stationary distribution is unique. In this case, it is given by sampling faces without replacement: explicitly, one sums over all permutations of the faces to obtain

$$\pi_C = \sum_{\substack{\sigma \in S_N : \\ C = F_{\sigma(1)} \cdots F_{\sigma(N)}}} \prod_{1 \leq p \leq N} \frac{w_{F_{\sigma(p)}}}{1 - \sum_{i<p} w_{F_{\sigma(i)}}}, \tag{2.5}$$

where $N = |\mathcal{F}|$, the number of faces. The complete list of eigenvalues is given as follows, where μ denotes the Möbius function of $L(\mathcal{A})$. For each $X \in L(\mathcal{A})$, let

$$\lambda_X = \sum_{F:\, |F| \geq X} w_F. \tag{2.6}$$

(Equivalently, $\lambda_X = \sum_{F \in \mathcal{F}(\mathcal{A}^X)} w_F$, by (2.1).)

Theorem 2.6 ([1, 3])**.** *For each $X \in L_p(\mathcal{A})$, for $0 \leq p \leq \ell$, the matrix K has an eigenspace of multiplicity $(-1)^p \mu(W, X)$ with eigenvalue λ_X.*

The main result here is a corresponding basis of eigenvectors for each eigenvalue, Theorem 3.6. For this, it is convenient to regard the distribution weights w_F as indeterminates as in [2], diagonalize, and then specialize afterwards: let $R = \mathbb{R}(w_F \colon F \in \mathcal{F})$, the fraction field of polynomials in the variables w_F. In doing so, we relax the condition that the weights sum to 1, so (2.5) needs to be adjusted accordingly. Let $q \in V(\mathcal{A})$ be the vector whose coordinate on chamber C is given by

$$q_C = \sum_{\substack{\sigma \in S_N \colon \\ C = F_{\sigma(1)} \cdots F_{\sigma(N)}}} \prod_{p=1}^{N} \Big(\sum_{i=p}^{N} w_{F_{\sigma(i)}} \Big)^{-1}. \tag{2.7}$$

Lemma 2.7. *For any arrangement $\mathcal{A}$, the vector q is a λ_W-eigenvector of K.*

Proof. We provide a direct calculation in lieu of adapting the corresponding result from [3]. For succinctness, let

$$f(x_1, \ldots, x_N) = \prod_{i=1}^{N} \Big(\sum_{i=p}^{N} x_i \Big)^{-1}$$

for any choice of x_i's, and abbreviate $f(\sigma) := f(w_{F_{\sigma(1)}}, \ldots, w_{F_{\sigma(N)}})$ for any permutation of the faces $\sigma \in S_N$. By clearing denominators, one may verify the identity

$$\begin{aligned} f(x_1, \ldots, x_N) + & f(x_2, x_1, x_3, \ldots, x_N) + \cdots \\ & + f(x_2, \ldots, x_N, x_1) = f(x_1, \ldots, x_N) \cdot \Big(\sum_{i=1}^{N} x_i \Big) / x_1. \end{aligned} \tag{2.8}$$

For $1 \leq i \leq N$, let σ_i denote the i-cycle $(1, 2, \ldots, i) \in S_N$. Then (2.8) states that

$$\sum_{i=1}^{N} f(\sigma \sigma_i^{-1}) = (\lambda_W / x_1) f(\sigma), \tag{2.9}$$

since $\lambda_W = \sum_{i=1}^{N} w_{F_i}$.

Now $Kq = \sum_{F\in\mathcal{F}} w_F q_C(FC)$. For each $C \in \mathcal{C}$, we compute:

$$\begin{aligned}
(Kq)_C &= \sum_{F,C'\colon C=FC'} w_F q_{C'} \\
&= \sum_{\substack{F\in\mathcal{F},\,\sigma\in S_N\colon\\ C=FF_{\sigma(1)}\cdots F_{\sigma(N)}}} w_F f(\sigma) \\
&= \sum_{\substack{\sigma\in S_N\colon\\ C=F_{\sigma(1)}\cdots F_{\sigma(N)}}} \sum_{i=1}^{N} w_{F_{\sigma(1)}} f(\sigma\sigma_i^{-1}),\ \text{by the deletion property, §2.1,} \\
&= \lambda_W q_C,\ \text{by (2.9).}
\end{aligned}$$

□

Remark 2.8. As written, the weights $\{w_F\}$ in (2.7) only admit specializations to nonzero real numbers. One may clear denominators to obtain a general polynomial expression for an eigenvector, however this is clumsy to write in general.

2.9. **Combinatorial Heaviside functions.** We will recover Theorem 2.6, together with eigenvectors, by exploiting some fundamental structural results of Varchenko and Gel′fand [10], which we briefly describe here. For more details, see [4].

The Varchenko-Gel′fand ring of $\mathcal{A}$ is defined additively to be simply the space of linear functionals on chambers, V^*. The ring structure is given by coordinatewise multiplication. The interest lies in the choice of generators: for each hyperplane $H \in \mathcal{A}$, let $x_H \in V^*$ be the function defined by

$$x_H(C) = \begin{cases} 1 & \text{if } C \text{ is on the positive side of } H; \\ 0 & \text{otherwise.} \end{cases}$$

For a set of hyperplanes $I \subseteq \mathcal{A}$, let x_I be the monomial $x_I = \prod_{H\in I} x_H$. (Since each x_H is idempotent, we only need to consider square-free monomials.) Let $1 \in V^*$ be the function given by $1(C) = 1$ for all chambers $C \in \mathcal{C}$.

In [10], it is shown that the Varchenko-Gel′fand ring admits a presentation much like the Orlik-Solomon algebra, with generators x_H and certain combinatorial relations: see [7] for an interpretation that compares the two. Unlike the Orlik-Solomon algebra, however, the relations amongst the generators are inhomogeneous, so it is useful to define a degree filtration by letting

$$P_pV^* = \{f \in V^*\colon f \text{ can be written as a polynomial in } x_H\text{'s of degree at most } p.\}$$

By [10, Theorem 1],

$$V^* = P_\ell V^* \supseteq P_{\ell-1}V^* \supseteq \cdots P_0 V^* \supsetneq 0,$$

with the function 1 spanning P_0. The filtration is "natural" in the sense that, if $\mathcal{B}$ is a subarrangement of $\mathcal{A}$, containment gives a map of chambers $\mathcal{C}(\mathcal{A}) \to \mathcal{C}(\mathcal{B})$. This induces a map $V(\mathcal{B})^* \to V(\mathcal{A})^*$, which is easily seen to preserve the degree filtration.

Now let $\operatorname{gr}_p V^* = P_pV^*/P_{p-1}V^*$, for $0 \leq p \leq \ell$. Let $b_p = (-1)^p \sum_{X \in L_p(\mathcal{A})} \mu(W, X)$, the pth Betti number of the arrangement. Then the degree filtration satisfies an analogue of Brieskorn's Lemma for arrangements:

Theorem 2.10 (Theorem 3, Corollaries 2,3 in [10])**.** *For $0 \leq p \leq \ell$, $\operatorname{gr}_p V^*$ is a free module, of rank equal to $b_p(\mathcal{A})$. More precisely, there are isomorphisms*

$$\operatorname{gr}_p V(\mathcal{A})^* \cong \bigoplus_{X \in L_p(\mathcal{A})} \operatorname{gr}_p V(\mathcal{A}_X)^*. \tag{2.10}$$

induced by the inclusion of the arrangement $\mathcal{A}_X$ into $\mathcal{A}$.

Example 2.11. Consider the arrangement of lines $\{x, y, x-y\}$ through the origin in $\mathbb{R}^2$, as in Figure 1. Call the lines H_1, H_2, H_3 in this order. Bases for $\operatorname{gr}_p V^*$ are:

$$\begin{array}{r|l} p=2 & x_{H_1}x_{H_2},\ x_{H_1}x_{H_3}, \\ p=1 & x_{H_i}\colon 1 \leq i \leq 3, \\ p=0 & 1. \end{array}$$

For example, the function $x_{H_1}x_{H_3}$ takes the value 1 on the chamber $x > 0$, $x > y$, and zero elsewhere.

2.12. **The dual filtration.** The degree filtration defines an orthogonal, decreasing filtration on the dual space, $V^{**} \cong V$: following [10], let

$$W^pV = \{v \in V\colon f(v) = 0 \text{ for all } f \in P_{p-1}V^*.\}$$

Then

$$V = W^0V \supseteq W^1V \supseteq \cdots \supseteq W^{\ell+1}V = 0.$$

If $\mathcal{B}$ is a subarrangement of $\mathcal{A}$ and $i\colon \mathcal{C}(\mathcal{A}) \to \mathcal{C}(\mathcal{B})$ is the induced map of chambers, the "natural" map $V(\mathcal{A}) \to V(\mathcal{B})$ extends i linearly. Dually, this map preserves the W-filtration.

Let $\operatorname{gr}^p V = W^pV/W^{p+1}V$, for $0 \leq p \leq \ell$. The dual version of Theorem 2.10 reads as follows:

Proposition 2.13. *For $0 \leq p \leq \ell$, we have $\operatorname{gr}_p V^* \cong (\operatorname{gr}^p V)^*$. So $\operatorname{gr}^p V$ is also free of rank b_p, and admits a decomposition*

$$\operatorname{gr}^p V(\mathcal{A}) \cong \bigoplus_{X \in L_p(\mathcal{A})} \operatorname{gr}^p V(\mathcal{A}_X). \tag{2.11}$$

Proof. We prove the first assertion, from which the rest follows directly. Let $e\colon V \to V^{**}$ be the natural isomorphism. If $x \in W^pV$, then $e(x)$ restricts to a map $P_pV^* \to R$ by $x(f) = f(x)$. If $e(x) = 0$, this means $f(x) = 0$ for all $f \in P_pV^*$, so $x \in W^{p+1}V$.

On the other hand, restriction of functions gives a surjective map $\operatorname{res}\colon (P_pV^*)^* \to (P_{p-1}V^*)^*$. Putting this together, we have shown that the sequence

$$0 \longrightarrow W^{p+1}V \longrightarrow W^pV \xrightarrow{e} (P_pV^*)^* \xrightarrow{\text{res}} (P_{p-1}V^*)^* \longrightarrow 0 \tag{2.12}$$

is exact except possibly at $(P_pV^*)^*$. For this, suppose ϕ is in the kernel of res. Let $x = e^{-1}(\phi)$. That ϕ restricts to zero means $f(x) = 0$ for all $f \in P_{p-1}V^*$, which is to say that $x \in W^pV$, and (2.12) is exact.

The cokernel of $W^{p+1}V \hookrightarrow W^pV$ is $\operatorname{gr}^p V$, by definition. Since $\operatorname{gr}_p V^*$ is free (Theorem 2.10), the kernel of the map res is $(\operatorname{gr}_p V^*)^*$. It follows that the restriction of e induces an isomorphism $\operatorname{gr}^p V \cong (\operatorname{gr}_p V^*)^*$. □

2.14. **Flag cochains.** The dual counterparts of the combinatorial Heaviside functions are the *flag cochains* of [10], which we reformulate slightly following [4]. First, for a ranked poset P, for $p \geq 0$ let $\operatorname{Flag}_p(P)$ be the set of p-flags: that is, chains $x_0{<}x_1{<}\cdots{<}x_p$ in P where x_i has rank i for $0 \leq i \leq p$. For a fixed arrangement $\mathcal{A}$, let $\mathrm{Fl}_p = \mathrm{Fl}_p(\mathcal{A})$ be the free abelian group on $\operatorname{Flag}_p(L(\mathcal{A}))$, modulo the relations

$$\sum_{Y\colon X_{i-1}<Y<X_{i+1}} (X_0{<}\cdots{<}X_{i-1}{<}Y{<}X_{i+1}\cdots{<}X_p),$$

for each flag $(X_0{<}\cdots{<}X_p)$ and index i, $0 < i < p$. The groups Fl_p are isomorphic to the homology groups of the complexified complement of $\mathcal{A}$: see [9]. In particular,

$$\mathrm{Fl}_p(\mathcal{A}) \cong \bigoplus_{X\in L_p(\mathcal{A})} \mathrm{Fl}_p(\mathcal{A}_X), \tag{2.13}$$

a dual formulation of Brieskorn's Lemma. The rank of the summand indexed by X is $|\mu(W, X)|$.

Let $\mathbf{F}$ be a flag in $\mathcal{F}(\mathcal{A})$: then $\mathbf{F} = (F_0{<}F_1{<}\cdots{<}F_p)$ for some p, where each F_i is a face of codimension i. Since we continue to assume that $\mathcal{A}$ is a central arrangement, each face F has an antipodally opposite face, which we denote by $\overline{F}$. Define an element of V using the following expression in the face algebra:

$$b(\mathbf{F}) = F_p(F_{p-1} - \overline{F_{p-1}})(F_{p-2} - \overline{F_{p-2}})\cdots(F_0 - \overline{F_0}). \tag{2.14}$$

Varchenko and Gel′fand [10] find that the flag cochains $b(\mathbf{F})$ span W^pV, for each p. In fact, they do so in a way compatible with the Brieskorn decompositions (2.11), (2.13). In order to indicate how this goes, we need some additional notation.

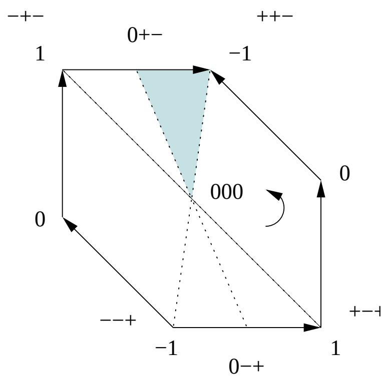
Example 2.15. For the arrangement of Example 2.11, let $\mathbf{F} = ({+}{+}{-}{<}0{+}{-}{<}000)$. Then $b(\mathbf{F}) = C_{++-} - C_{-+-} - C_{+-+} + C_{--+}$.

For any $\mathbf{F} \in \operatorname{Flag}_p(\mathcal{F})$, let $|\mathbf{F}|$ denote the flag $\mathbf{X}$ in the intersection lattice with $X_i = |F_i|$. Regard F_p as a chamber of $\mathcal{A}^{X_p}$, in order to define a map

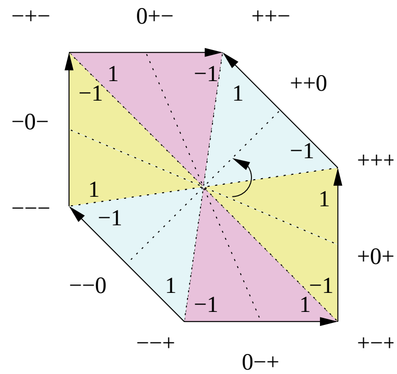
$$f\colon \operatorname{Flag}_p(\mathcal{F}(\mathcal{A})) \to \coprod_{X\in L_p(\mathcal{A})} \operatorname{Flag}_p(L(\mathcal{A}_X)) \times \mathcal{C}(\mathcal{A}^X)$$

by $f(\mathbf{F}) = (|\mathbf{F}|, F_p)$. Consider the fibres of f. If $f(\mathbf{F}) = (\mathbf{X}, F)$, then $F_p = F$. There are two possibilities for F_{p-1}, however, obtained by moving away from F inside X_{p-1} in either of two directions. Once F_{p-1} is chosen, there are two possiblities for F_{p-2}, and so on. By inspecting (2.14), one finds that flag cochains $b(\mathbf{F})$ differ at most by a sign on flags in the same fibre of the map f.

In order to reconcile the choice of signs, choose an orientation of each face of the zonotope $Z_{\mathcal{A}}$ so that parallel faces have the same orientation, but arbitrarily otherwise. (This is equivalent to choosing a coorientation of each element of $L(\mathcal{A})$, as in [10], but easier to draw.) Then for each covering relation $F < G$ in $\mathcal{F}(\mathcal{A})$, let $\varepsilon(F, G) =$



(a) A flag cochain



(b) A relation in $\mathrm{Fl}_2(\mathcal{A})$

FIGURE 3. Flag cochains

± 1 according to whether or not the orientations on F and G agree. Let $\varepsilon(\mathbf{F}) = \prod_{i=0}^{p-1} \varepsilon(F_i, F_{i+1})$.

The last result we need to recall is the dual formulation of [10, Theorem 8].

Theorem 2.16. *For $0 \leq p \leq \ell$, there is a well-defined map depending on the choices of orientations,*

$$\pi_p \colon W^p V(\mathcal{A}) \to \mathrm{Fl}_p(\mathcal{A}), \tag{2.15}$$

for which $\pi_p(b(\mathbf{F})) = \varepsilon(\mathbf{F})|\mathbf{F}|$, for all flags $\mathbf{F} \in \mathrm{Flag}_p(\mathcal{F})$. The kernel of π_p is $W^{p+1}V(\mathcal{A})$.

There is also a map in the other direction, given by [11, Theorem 18.3.3]:

Proposition 2.17 ([11])**.** *For each p, the map*

$$\phi_p \colon \bigoplus_{X \in L_p(\mathcal{A})} \mathrm{Fl}_p(L(\mathcal{A}_X)) \otimes_R V(\mathcal{A}^X) \to W^p V(\mathcal{A}) \tag{2.16}$$

given by sending $\mathbf{X} \otimes F$ to $\varepsilon(\mathbf{F})b(\mathbf{F})$ is well-defined, where $\mathbf{F}$ is any flag with $f(\mathbf{F}) = (\mathbf{X}, F)$.

Example 2.18 (Example 2.15, continued)**.** Orient the faces of the zonotope as shown in Figure 3. For the flag $\mathbf{X} = (\mathbb{R}^2, H_1, 0)$, the only choice for F is the face 000. Picking $\mathbf{F} = ({+}{+}{-}{<}0{+}{-}{<}000)$, we see $\varepsilon(\mathbf{F}) = -1$, and coordinates of $\phi(\mathbf{X} \otimes C_{000})$ are as indicated. The simplex corresponding to $\mathbf{F}$ in the barycentric subdivision of $Z_{\mathcal{A}}$ is shaded in Figure 3(a).

For $\mathbf{X} = (\mathbb{R}^2, H_1)$, there are two choices for F. We find $\phi(\mathbf{X} \otimes C_{0-+}) = C_{+-+} - C_{--+}$ and $\phi(\mathbf{X} \otimes C_{0+-}) = C_{++-} - C_{-+-}$.

3. EIGENVECTORS FOR THE RANDOM WALK

3.1. **The face algebra action on flag cochains.** We recall that $A = R[\mathcal{F}]$ denoted the face algebra. The domain of the map ϕ_p in (2.16) has the structure of a left A-module by the action of A on each $V(\mathcal{A}^X)$. We see in this section that the codomain

of ϕ_p is also an A-module (Corollary 3.3), and ϕ_p is an A-module homomorphism (Lemma 3.4).

First, consider applying the face product to each element of a flag $\mathbf{F} \in \mathrm{Flag}_p(\mathcal{F})$. Clearly for each $F \in \mathcal{F}$, we have $FF_0 \leq \cdots \leq FF_p$. By [4, Lemma 3.2], however, this is a flag if and only if $|F| \geq |F_p|$; otherwise, $FF_k = FF_{k+1}$ for some k with $0 \leq k \leq p-1$. As usual, we regard the faces F with $|F| \geq |F_p|$ as chambers of $\mathcal{A}^{|F_p|}$. In this case, let $F\mathbf{F}$ denote the flag $FF_0 < \cdots < FF_p$.

Proposition 3.2. *For each $F \in \mathcal{F}$ and $\mathbf{F} \in \mathrm{Flag}_p(\mathcal{F})$, we have*

$$Fb(\mathbf{F}) = \begin{cases} b(F\mathbf{F}) & \text{if } |F| \geq |F_p|; \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By the deletion property,

$$\begin{aligned} Fb(\mathbf{F}) &= FF_p(F_{p-1} - \overline{F_{p-1}})(F_{p-2} - \overline{F_{p-2}}) \cdots (F_0 - \overline{F_0}) \\ &= FF_p(FF_{p-1} - F\overline{F_{p-1}})(FF_{p-2} - F\overline{F_{p-2}}) \cdots (FF_0 - F\overline{F_0}). \end{aligned}$$

If $|F| \geq |F_p|$, then $|F| \geq |F_i|$ for each i, so $F\overline{F_i} = \overline{FF_i}$ for each i, and we obtain $b(F\mathbf{F})$.

Otherwise, $FF_k = FF_{k+1}$ for some k. Since $F\overline{F} = F$ for any F, it is easy to check that $(FF_k - F\overline{F_k})(FF_k - F\overline{F_k}) = 0$, from which it follows that $Fb(\mathbf{F}) = 0$ as well. $\square$

Since W^pV is spanned by vectors $b(\mathbf{F})$ for $\mathbf{F} \in \mathrm{Flag}_p(\mathcal{F})$, we see the filtration is compatible with the face product. That is,

Corollary 3.3. *For $0 \leq p \leq \ell$, W^pV is an A-submodule of V.*

Lemma 3.4. *The map ϕ of* (2.16) *is an A-module homomorphism.*

Proof. For $\mathbf{X} \otimes F \in \mathrm{Fl}_p(\mathcal{A}_X) \otimes V(\mathcal{A}^X)$, choose a flag $\mathbf{F}$ with $|\mathbf{F}| = \mathbf{X}$ and $F_p = F$. Recall that we chose the orientations in $Z_{\mathcal{A}}$ to agree on parallel faces. Note the face $\widehat{GF_i}$ is a translate of $\hat{F}_i$ (Proposition 2.4), for all $0 \leq i \leq p$ and $|G| \geq |F_p|$, so their orientations agree. It follows that $\varepsilon(\mathbf{F}) = \varepsilon(G\mathbf{F})$ for all faces G with $|G| \geq X$. So for $G \in \mathcal{C}(\mathcal{A}^X)$,

$$\begin{aligned} \phi(G(\mathbf{X} \otimes F)) &= \varepsilon(G\mathbf{F})b(G\mathbf{F}) \\ &= \varepsilon(\mathbf{F})Gb(\mathbf{F}), \\ &= G\phi(\mathbf{X} \otimes F) \end{aligned}$$

using Proposition 3.2 at the second step. On the other hand, if $|G| \not\geq X$, both sides are zero, by (2.3) and Proposition 3.2. $\square$

3.5. **The main result.** Now we are able to state and prove a description of each eigenspace of the BHR random walk's transition matrix K. The main idea is to use the stationary distribution (2.5) on each the arrangement $\mathcal{A}^X$, for each X, together with Lemma 3.4.

Recall that eigenvalues of K were indexed by subspaces $X \in L(\mathcal{A})$, from (2.6). For each $X \in L(\mathcal{A})$, then let $q^X \in V(\mathcal{A}^X)$ be the eigenvector (2.7) for the arrangement $\mathcal{A}^X$, a vector over the ring R: by Lemma 2.7,

$$K_{\mathcal{A}^X} \cdot q^X = \lambda_X q^X. \tag{3.1}$$

For each $C \in \mathcal{C}(\mathcal{A}^X)$, let q^X_C denote the Cth coordinate of q^X.

For each $X \in L_p(\mathcal{A})$, define a map $\psi_X \colon \operatorname{Fl}_p(\mathcal{A}_X) \to W^p V(\mathcal{A}) \subseteq V(\mathcal{A})$ by letting

$$\psi_X(\mathbf{X}) = \sum_{C \in \mathcal{C}(\mathcal{A}^X)} q^X_C \phi(\mathbf{X} \otimes C). \tag{3.2}$$

Let $\psi \colon \operatorname{Fl}(\mathcal{A}) \to V$ be given on $\operatorname{Fl}_p(\mathcal{A})$ by composing the isomorphism (2.13) with the maps ψ_X.

Theorem 3.6. *The map ψ_X is one-to-one, and its image is the eigenspace of $K_\mathcal{A}$ with eigenvalue λ_X.*

In other words, for each flag $\mathbf{X}$ ending at X, and each face F with $|F| = X$, choose any flag $\mathbf{F} \in \operatorname{Flag}(\mathcal{F})$ with $|\mathbf{F}| = \mathbf{X}$, ending at F. Then the vector

$$\sum_{F \colon |F| = X} \varepsilon(\mathbf{F}) q^X_F b(\mathbf{F})$$

is an eigenvector for K with eigenvalue λ_X. Moreover, we see now that the Varchenko-Gel′fand dual filtration (§2.12) can be used to keep track of the eigenspace multiplicities (Theorem 2.6):

Corollary 3.7. *The map ψ is an isomorphism, taking the Brieskorn decomposition* (2.13) *of* $\operatorname{Fl}(\mathcal{A})$ *to the eigenspace decomposition of V.*

Proof of Theorem 3.6. First check $\psi_X(\mathbf{X})$ is an eigenvector, for any flag $\mathbf{X}$. We have

$$\begin{aligned} K \cdot \psi(\mathbf{X}) &= \sum_{F \in \mathcal{F}(\mathcal{A})} w_F F \sum_{C \in \mathcal{C}(\mathcal{A}^X)} q^X_C \phi(\mathbf{X} \otimes C) \\ &= \sum_{F \in \mathcal{F}(\mathcal{A}^X)} w_F \sum_{C \in \mathcal{C}(\mathcal{A}^X)} q^X_C \phi(\mathbf{X} \otimes FC) \\ &= \lambda_X \psi(\mathbf{X}), \end{aligned}$$

using first Lemma 3.4 then (3.1).

To see that the kernel of ψ_X is zero, consider the two short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & W^{p+1}V(\mathcal{A}) \arrow[r] \arrow[d, "i^*"'] & W^p V(\mathcal{A}) \arrow[r, "\pi_p"] \arrow[d, "i^*"'] & \operatorname{Fl}_p(\mathcal{A}) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & W^{p+1}V(\mathcal{A}_X) \arrow[r] & W^p V(\mathcal{A}_X) \arrow[r, "\pi_p"] & \operatorname{Fl}_p(\mathcal{A}_X) \arrow[r] \arrow[ul, dotted, "\psi_X"'] & 0, \end{tikzcd}$$

where the rows are given by (2.15), and the vertical maps are the natural ones (§2.12). Note that $i^* b(\mathbf{F}) = i^* b(\mathbf{F}')$ for any two flags $\mathbf{F}$ and $\mathbf{F}' \in \operatorname{Flag}_p(\mathcal{F}(\mathcal{A}_X))$. It follows

that $i^* \circ \phi(\mathbf{X} \otimes C) = i^* \circ \phi(\mathbf{X} \otimes C')$ for any $C, C' \in \mathcal{C}(\mathcal{A}^X)$, so the composite

$$\begin{aligned} \pi_p \circ i^* \circ \psi_X(\mathbf{X}) &= \pi_p\Big(\big(\sum_{C \in \mathcal{C}(\mathcal{A}^X)} q_C^X\big) i^* \phi(\mathbf{X} \otimes C)\Big) \\ &= \big(\sum_{C \in \mathcal{C}(\mathcal{A}^X)} p_C^X\big)\mathbf{X}. \end{aligned}$$

Since the vector $q^X \in V(\mathcal{A}^X)$ is a rescaling of a probability distribution, the sum of its coordinates must be nonzero. Since our coefficient ring R is a domain, this means that the composite has zero kernel, so ψ_X is one-to-one. □

Example 3.8 (Example 2.18, continued)**.** Here is a basis of eigenvectors in the case of three lines in the plane given by Theorem 3.6. The $\lambda_{\mathbb{R}^2}$-eigenvector is provided by q of (2.7): if we specialize the weights to a probability distribution, recall $1 = \lambda_{\mathbb{R}^2}$, and the eigenvector is the stationary distribution (2.5).

For an arrangement of one point in a line, the vector (2.7) equals

$$\frac{w_+ + w_0 + w_-}{w_0 w_+ w_- (w_+ + w_-)} \cdot (w_+, w_-), \tag{3.3}$$

ordering the chambers with + first. Let $\mathbf{X} = (\mathbb{R}^2, H_1)$. Then $\Phi_{H_1}(\mathbf{X})$ is a unit multiple of $w_{0+-}\phi(\mathbf{X} \otimes C_{0+-}) + w_{0-+}\phi(\mathbf{X} \otimes C_{0-+})$, as in (3.3). Using the calculation in Example 2.18 gives the vector shown in Figure 4.

We order the basis of V counterclockwise, starting with the chamber $+++$. For clarity, the weights of the codimension-0 and -1 faces are relabelled $w_0, \ldots, w_6$ as shown in Figure 4. Then the eigenvectors given by Theorem 3.6 are

$\mathbf{X}$	$\Psi(\mathbf{X})$ proportional to	λ
$(\mathbb{R}^2)$	p, above	$\sum_{i=0}^6 w_i$
$(\mathbb{R}^2, H_1)$	$(0, w_2, -w_2, 0, -w_5, w_5)$	$w_0 + w_2 + w_5$
$(\mathbb{R}^2, H_2)$	$(w_6, 0, w_3, -w_3, 0, -w_6)$	$w_0 + w_3 + w_6$
$(\mathbb{R}^2, H_3)$	$(-w_1, w_1, 0, w_4, -w_4, 0)$	$w_0 + w_1 + w_4$
$(\mathbb{R}^2, H_1, 0)$	$(0, -1, 1, 0, -1, 1)$	w_0
$(\mathbb{R}^2, H_2, 0)$	$(1, 0, -1, 1, 0, -1)$	w_0

Remark 3.9. The results of [10] on which Theorem 3.6 is based can likely be generalized to arbitrary oriented matroids without change, in which case the results here would generalize in the same way: see the discussion in [3].

In [2], Brown shows that many of the BHR random walk's properties (such as diagonalizability) can also be generalized by replacing the face algebra with any semigroup with the left-regular band property. The same paper describes idempotents for irreducible modules, which implicitly give eigenvectors for the generalized random walk. However, that description is somewhat more complicated than the one here. Recent work of Saliola [8] gives a quite different description of the eigenvectors of the BHR random walk, using methods that hold for any left-regular band. His construction is complementary to the one given here. The cost of working with the general setting of [2] is apparently no longer to have an eigenbasis with such convenient labelling as

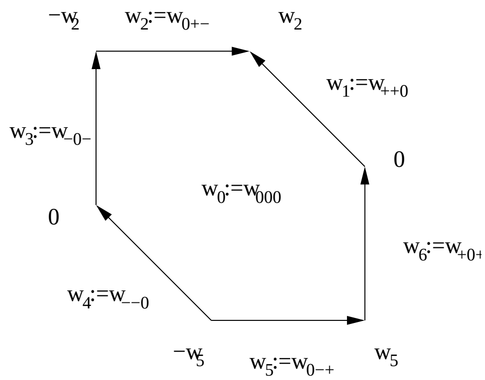


FIGURE 4. $\Psi_{H_1}(\mathbb{R}^2, H_1)$ in Example 3.8

non-broken circuits. It would be interesting to know, then, if the main results of [10] would admit left-regular band generalizations.

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